NPS-MA-91-011

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ON SINGULAR SEMILINEAR
ELLIPTIC EQUATIONS

by

Aihua W. Shaker

A Technical Report for Period

October 1990 - April 1991

Prepared for: Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93943

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared in conjunction with research conducted for the Naval Postgraduate School and funded by the Naval Postgraduate School.

Prepared by:

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION	Unclassified
1b RESTRICTIVE MARKINGS	
2a SECURITY CLASSIFICATION AUTHORITY	
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
3 DISTRIBUTION/AVAILABILITY OF REPORT	Approved for public release; distribution unlimited
4 PERFORMING ORGANIZATION REPORT NUMBER(S)	NPS-MA-91-011
5 MONITORING ORGANIZATION REPORT NUMBER(S)	NPS-MA-91-011
6a NAME OF PERFORMING ORGANIZATION	Naval Postgraduate School
6b OFFICE SYMBOL (If applicable)	MA
7a NAME OF MONITORING ORGANIZATION	Naval Postgraduate School
6c ADDRESS (City, State, and ZIP Code)	Monterey, CA 93943
7b ADDRESS (City, State, and ZIP Code)	Monterey, CA 93943
8a NAME OF FUNDING/SPONSORING ORGANIZATION	Naval Postgraduate School
8b OFFICE SYMBOL (If applicable)	MA
9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER	O&MN Direct Funding
8c ADDRESS (City, State, and ZIP Code)	Monterey, CA 93943

10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO	TASK NO	WORK UNIT ACCESSION NO.

11 TITLE (Include Security Classification)

ON SINGULAR SEMILINEAR ELLIPTIC EQUATIONS Unclassified

12 PERSONAL AUTHOR(S)

Aihua W. Shaker

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Technical Report	FROM 10/90 TO 04/91	91/05/08	9

16 SUPPLEMENTARY NOTATION

17 COSATI CODES		
FIELD	GROUP	SUB-GROUP

18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

Singular, Semilunar, Elliptic, Entire solutions, Asymptotic behavior

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

Abstract: - For the semilinear elliptic equation $\Delta u + p(x)u^{-\gamma} = 0,\ x \in R^n$, $n \geq 3,\ \gamma > 0$, we show via the barrier method the existence of a positive entire solution behaving like $|x|^{2-n}$ near ∞.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT

☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS

21 ABSTRACT SECURITY CLASSIFICATION

UNCLASSIFIED

22a NAME OF RESPONSIBLE INDIVIDUAL	22b TELEPHONE (Include Area Code)	22c OFFICE SYMBOL
Aihua Shaker	408-646-2089	MA/Sa

DD Form 1473, JUN 86 Previous editions are obsolete SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603 UNCLASSIFIED

On Singular Semilinear Elliptic Equations

Aihua W. Shaker
Department of Mathematics
Naval Postgraduate School
Monterey, CA93943

May 8, 1991

Abstract: - For the semilinear elliptic equation $\Delta u + p(x)u^{-\gamma} = 0$, $x \in R^n$, $n \geq 3$, $\gamma > 0$, we show via the barrier method the existence of a positive entire solution behaving like $|x|^{2-n}$ near ∞.

1 Introduction

We study the singular semilinear elliptic equation

$$\Delta u + p(x)u^{-\gamma} = 0 \tag{1}$$

in R^n. This type of equation arises in the boundary layer theory of viscous fluids [3,4]. From the results of Fulks and Maybee [8], Crandall, Rabinowith, and Tartar [5], Gomes [9], and recently Lazer and McKenna [14], it follows that (1) has a unique classical solution within a bounded domain Ω, where $p(x)$ is a sufficiently regular function which is positive on $\bar{\Omega}$. Kusano and Swanson [12] gave the existence proof on exterior domains. As for the existence of entire solutions, not much is known. Edelson [7], Kusano and Swanson [13] have been able to show the existence of entire solutions of (1) with $\gamma \in (0,1)$, and $p(x)$ sufficiently regular. In this paper we show via the upper and lower solution method, which is also referred to as the barrier method, that (1) has a bounded positive entire solution vanishing at ∞ in R^n for $n \geq 3$ and all $\gamma > 0$.

The author learned after this paper was finished that a similar result was given earlier by R. Dalmasso [6], but by a different approach.

2 Preliminaries

We first state the theorem by Kusano and Swanson [13] for the case $0 < \gamma < 1$.

Lemma 1. *Equation (1) has an entire bounded positive solution $u(x)$ in R^n for $n \geq 3$,and $|x|^{n-2}u(x)$ is bounded and bounded away from zero near ∞ if $p(x)$ satisfies the following conditions:*
(H1) $p(x) \in C^{\alpha}_{loc}(R^n)$, $n \geq 3$, $p(x) > 0$, $x \in R^n \setminus \{0\}$,
(H2) $\exists C > 0$, *such that* $C\phi(|x|) \leq p(x) \leq \phi(|x|)$, $\phi(x) = max_{|x|=t}p(x)$, $0 \leq t \leq \infty$,
(H3) $\int_1^\infty t^{n-1+\gamma(n-2)}\phi(t)dt < \infty$.

The term "entire" has often been used for solutions of equation (1) in R^n. To avoid confusion with the traditional definition for entire functions, we use the term "$C^{2+\alpha}$-entire". A $C^{2+\alpha}$-entire solution of (1) is defined to be a function $u(x) \in C^{2+\alpha}_{loc}(R^n)$ that satisfies (1) pointwise in R^n.

The method that we shall be using heavily in our proof is the so-called **barrier method**, or **upper-lower solution** method.

We consider the elliptic boundary value problem

$$(2) \qquad \begin{cases} Lu + f(x,u) = 0 & \text{in } D \\ Bu = a\partial u/\partial\nu + bu = g & \text{on } \partial D \end{cases}$$

where D is a smoothly bounded domain in R^N and $\nu = (\nu_1, \cdots, \nu_n)$ is a smoothly varying outward normal vector field on ∂D which is of class $C^{2+\alpha}$, while a and b are positive constants. We also assume that $f \in C^\alpha$ and that g has an extension $\hat{g}$ to the interior of D such that $\hat{g} \in C^{2+\alpha}$.

An **upper solution** to the above problem is a function ϕ satisfying

$$\begin{cases} L\phi + f(x,\phi) \leq 0 & \text{in } D \\ B\phi \geq g & \text{on } \partial D. \end{cases}$$

A **lower solution** to the above problem is a function ψ satisfying

$$\begin{cases} L\psi + f(x,\psi) \geq 0 & \text{in } D \\ B\psi \leq g & \text{on } \partial D. \end{cases}$$

We assume that $\partial D, f, g$, and the coefficients of L are smooth in what follows.

Lemma 2. *(Theorem 2.3.1 of [16]) Let ϕ be an upper solution and ψ a lower solution with $\psi \leq \phi$ on D. Then there exists a solution u to the above boundary value problem with $\psi \leq u \leq \phi$.*

We consider the following example:

$$\begin{cases} u'' + \lambda u - u^3 = 0 & x \in (0, \pi) \\ u = 0 & x = 0, \pi \end{cases}$$

By the above theorem, if $\lambda > 1$, then the problem has at least three solutions.

Actually, $\underline{u} = \epsilon \sin x$ with ϵ small is a lower solution, and $\bar{u} = Rx^{1/2}$ with R large is an upper solution. Therefore there exists a solution u such that $\underline{u} \leq u \leq \bar{u}$ in $(0, \pi)$. Clearly $-u$ and 0 are also solutions to this problem.

The following lemma on the barrier method for $D = R^n$ is due to Ni [15] in 1982. A special case was proved earlier by Ako and Kusano [1] in 1964. The proof is standard. Using the well known result on the upper-lower solution approach in bounded regions (see Sattinger [16]), we first solve the equation

$$Lu + F(x, u) = 0$$

on B_R. Then by letting, $R \to \infty$, we obtain a solution on R^n by a diagonal process.

Lemma 3. *Let $u_1 \geq u_2$ in R^n be such that*

$$\begin{cases} Lu_1 + f(x, u_1) \leq 0 \\ Lu_2 + f(x, u_2) \geq 0 \end{cases} \tag{3}$$

where f is locally Hölder continuous in (x, u) and locally Lipschitz in u, and L is an elliptic operator of second order. Then there exists a solution u of $Lu + f(x, u) = 0$ with $u_1 \geq u \geq u_2$.

3 Main Result

Theorem 1. *Under the same conditions as given in Lemma 1, the equation (1) has a $C^{2+\alpha}$-entire positive solution in $R^N, N \geq 3$, vanishing at ∞ at the rate of at least $|x|^{q(N-2)}$ with some $q \in (0, 1)$ for any $\gamma > 0$.*

The difficulty in constructing the proof is to find an appropriate upper solution for equation (1). In order to use the barrier method we first study the nonsingular equation

$$\Delta u + p(x)[\delta + u]^{-\gamma} = 0.$$

For each fixed γ there corresponds a solution $u_\gamma(x)$. Letting $\gamma \to \infty$, we show that the limiting function is the desired solution.

Proof: By Lemma 1, for $\gamma = \gamma_1 \in (0,1)$, equation (1) has a $C^{2+\alpha}$-entire positive solution $u_1(x)$ in R^n, $n \geq 3$, vanishing at ∞ at the rate r^{2-n}. We claim that $\bar{u} = cu_1^q$ is an upper solution of the equation (1) for $\gamma \geq 1$, where

$$q < \frac{1+\gamma_1}{1+\gamma} < 1,$$

$$c > \left(\frac{M^{1+\gamma_1-q(1+\gamma)}}{q}\right)^{\frac{1}{1+\gamma}}, \quad M = max_{x\in R^n}|u(x)|.$$

In fact:

$$\begin{aligned}
&\Delta \bar{u} + \frac{p(x)}{\bar{u}^\gamma} \\
&= cq(q-1)u^{q-2}|\nabla u|^2 - cqu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\
&\leq -cqu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma q} \\
&= \frac{p(x)}{c^\gamma u^{\gamma q}}\left(1 - \frac{c^{1+\gamma}q}{u^{1+\gamma_1-q(1+\gamma)}}\right) \\
&\leq \frac{p(x)}{c^\gamma u^\gamma}\left(1 - \left(\frac{M^{1+\gamma_1-q(1+\gamma)}}{q}\right)\frac{q}{u^{1+\gamma_1-q(1+\gamma)}}\right) \\
&\leq \frac{p(x)}{c^\gamma u^\gamma}(1-1) = 0.
\end{aligned}$$

Let δ be a fixed positive number. We then observe that $\bar{u}$ is an upper solution of the equation

$$\Delta u(x) + p(x)[u(x)+\delta]^{-\gamma} = 0, \; x \in R^n. \tag{4}$$

$\underline{u} = 0$ is a lower solution of (4). Since $\bar{u} = cu_1^q > 0$, $\bar{u} \geq \underline{u}$ in R^n. By Lemma 2, (4) has a solution u such that $\underline{u} \leq u \leq \bar{u}$.

For $\hat{\delta} < \delta$, u is a lower solution of (4) with $\delta = \hat{\delta}$. Lemma 2 then implies that (4) has a solution $\hat{u}$ for $\delta = \hat{\delta}$ such that $\underline{u} \leq \hat{u} \leq \bar{u}$.

Let $\{\delta_n\}_1^\infty$ be a sequence of strictly decreasing positive numbers, and let $u_n(x)$ be a smooth positive solution of (4) when $\delta = \delta_n$. From the construction of our lower solutions, it is clear that $u_n(x) \geq u_{n-1}(x)$ for all n. So $\lim_{n\to\infty} u_n(x) = u(x)$ exists for all $x \in R^n$ and

$$\underline{u} \leq u \leq \bar{u} \tag{5}$$

for $x \in R^n$.

We can now assert that $u \in C^{2+\alpha}(R^n)$ and that

$$\Delta u + p(x)u^{-\gamma} = 0 \tag{6}$$

for $x \in R^n$. This follows from more or less standard arguments.

Let $x_o \in R^n$ and $r > 0$. We consider the ball of radius r centered at x_o, $B(x_o, r)$ in R^n. Let Ψ be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/2)}$ and equal to 0 off $B(x_o, r)$. We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n$$

for $n \geq 1$, where p_n is a term whose L^∞ norm is bounded independently of n. Therefore for $n \geq 1$ we have

$$\Psi u_n \Delta(\Psi u_n) = \sum_{j=1}^{N} b_{nj} \frac{\partial(\Psi u_n)}{\partial x_j} + q_n,$$

where $b_{nj}, j = 1, \cdots, n$ and q_n are terms bounded independently of n for $n \geq 1$. Integrating the above equation, we have that there exist constants $c_1 > 0$ and $c_2 > 0$ independent of n such that

$$\int_{B(x_o,r)} |\nabla u_n|^2 dx \leq c_1 \left(\int_{B(x_o,r)} |\nabla u_n|^2 dx \right)^{1/2} + c_2.$$

From this, it follows that the $L^2(B(x_o, r))$-norm of $|\nabla \Psi u_n|$ is bounded independently of n. Hence, the $L^2(B(x_o, r/2))$-norm of $|\nabla u_n|$ is bounded independently of n. Let Ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/4)}$ and equal to 0 iff $B(x_o, r/2)$. We have for $n \geq 1$,

$$\Delta(\Psi_1 u_n) = 2\nabla\Psi_1 \cdot \nabla u_n + p_{1n},$$

where p_{1n} is a term whose $L^\infty(B(x_o, r/2))$- norm is bounded independently of n. From standard elliptic theory, the $W^{2,2}(B(x_o, r/2))$-norm of $\Psi_1 u_n$ is also bounded independently of n and hence, the $W^{2,2}(B(x_o, r/4))$-norm of u_n is bounded independently of n. Since the $W^{1,2}(B(x_o, r/4))$-norm of the components of ∇u_n are bounded independently of n, it follows from the Sobolev embedding theorem that if $q = 2n/(n-2) > 2$ for $n > 2$ and in addition if $q > 2$ is arbitrary for $n \leq 2$, then the $L^q(B(x_o, r/4))$-norm of $|u_n|$ is bounded independently of n. Let Ψ_2 be a C^∞ function which is equal to 1 on $\overline{B(x_o, r/8)}$ and equal to 0 iff $B(x_o, r/4)$. We have for $n \geq 1$,

$$\Delta(\Psi_2 u_n) = 2\nabla\Psi_2 \cdot \nabla u_n + p_{2n},$$

where p_{2n} is a term whose $L^\infty(B(x_o, r/4))$- norm is bounded independently of n. Since the right hand side of the above equation is bounded in $L^q(B(x_o, r/4))$ independently of n, the $W^{2,q}(B(x_o, r/4))$-norm of $\Psi_2 u_n$ is also bounded independently of n. Hence, the $W^{2,q}(B(x_o, r/8))$-norm of u_n is bounded independently of n. Continuing this line of reasoning, after a finite number of steps, we find a number $r_1 > 0$ and $q_1 > n/(1-\alpha)$ such that the $W^{2,q_1}(B(x_o, r_1))$-norm of u_n is bounded independently of n. Hence, there is a subsequence of $\{u_n\}_1^\infty$, which we may assume is the sequence itself, which converges in $C^{1+\alpha}\overline{(B(x_o, r_1))}$. If θ is a C^∞ function which is equal to 1 on $\overline{(B(x_o, r_1/2))}$ and 0 off $B(x_o, r_1)$, then

$$\Delta(\theta u_n) = 2\nabla\theta \cdot \nabla u_n + \hat{p_n}, \text{ where } \hat{p_n} = \theta\Delta u_n + u_n\Delta\theta.$$

The right-hand side of the above equation converges in $C^\alpha\overline{(B(x_o, r_1))}$. Hence by Schauder theory, $\{\theta u_n\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_o, r_1))}$ and thus $\{u_n\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_o, r_1/2))}$. Since x_o was arbitrary, this shows that $u \in C^{2+\alpha}(R^n)$. Clearly (6) holds.

4 Some remarks

Remark 1:

For $n = 1$, the properties of positive solutions of equation (1) have been studied by Taliaferro [17], and Gatica [10]. For $n = 2$, no entire positive solution of equation (1) exists regardless of its asymptotic behavior at ∞ (see [13]).

Remark 2:

It is observed by Callegari, Friedman and Nachman [2], [3,4] that if the partial differential equations describing the boundary layer behind a rarefaction or shock wave (with viscosity proportional to the temperature) traveling down, and perpendicular to, a flat plate are written in terms of a stream function and a similarity variable the following Blasius-type equation emerges [18].

$$f'''(\eta) + f(\eta)f''(\eta) = 0,$$

where

$$f(0) = 0, \;\; f'(0) = K, \;\; f'(\infty) = 1.$$

Here, $0 < K < 1$, for rarefaction waves and, $1 < K < 6$, for shock waves. ($K = 0$ corresponds to the classical Blasius problem.) Adopting the Crocco variables

$$x = f'(\eta), \;\; g = f''(\eta)$$

results in the system

$$gg'' + x = 0,$$

$$g'(K) = 0, \;\; g(1) = 0,$$

which falls into the class of equation discussed in this paper.

References

[1] K. Ako and T. Kusano, On bounded solutions of second order elliptic differential equations, J. Fac. Sci. Univ. Tokyo (Sect. I) 11 (1964), 29-37.

[2] A. J. Callegari and M. B. Friedman, An analytical solution of a nonlinear, singular boundary value problem in the theory of viscous fluids, J. Math. Anal. Appl. 21 (1968) 510-529.

[3] A. J. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math., 38 (1980), 275-281.

[4] A. J. Callegari and A. Nachman, Some singular, nonlinear equations arising in boundary layer theory, J. Math. Anal. Appl., 64 (1978),96-105.

[5] M. G. Crandall, P. H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Part. Diff. Eq. 2(2) (1977) 193-222.

[6] R. Dalmasso, Solutions d'equations elliptiques semi-lineaires singulieres, Annali di matematica pura et applicata serie quarta v.153 n4 1988 191-201.

[7] A. Edelson, Entire solutions of singular elliptic equations, J. Math. Anal. Appl. 139, 523-532 (1989).

[8] W. Fulks, L. S. Maybee, A Singular non-linear equation. Osaka Math. J. 12 (1960) 1-19.

[9] S. M. Gomes, On a singular nonlinear elliptic problem, Siam J. Math. Anal. 17, 6(1986) 1359-1369.

[10] J. A. Gatica, Vladimir Oliker, Singular nonlinear boundary value problems for second-order ordinary differential equations. J. Diff. Eq. 79, 62-78 (1989).

[11] C. E. Kenig and W. M. Ni, An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry, Amer. J. Math, 106 (1984), 689-702.

[12] T. Kusano and A. Swanson, Asymptotic properties of semilinear elliptic equations, Funkcial. Ekvac., 26 (1983), 115-129.

[13] T. Kusano and A. Swanson, Entire positive solutions of singular semilinear elliptic equations, Japan. J. Math. Vol. 11, No. 1, 1985.

[14] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary value problem. To appear.

[15] W. M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalization and application in geometry, Indiana Univ. Math. J. 31 (1982), 493-529.

[16] D. H. Sattinger, Topics in stability and bifurcation theory, Springer-Verlag, Berlin · Heidelberg · New York 1973.

[17] S. Taliaferro, On the positive solutions of $y'' + \phi(t)y^{-\lambda} = 0$, Nonlinear Anal., 2 (1978), 437-493.

[18] P. A. Thompson, Compressible-Fluid Dynamics, pp. 502-514, McGraw-Hill, New York, 1972.

DISTRIBUTION LIST

Director Defense Tech. Information Center Cameron Station Alexandria, VA 22314	(2)
Director of Research Administration Code 012 Naval Postgraduate School Monterey, CA 93943	(1)
Library Code 0142 Naval Postgraduate School Monterey, CA 93943	(2)
Professor Harold Fredricksen Department of Mathematics Naval Postgraduate School Monterey, CA 93943	(1)
Jame Cruz Department of Mathematics University of Virginia Charlottesville, VA 22903	
Richard Shaker Department of Mathematics University of Virginia Charlottesville, VA 22903	
Kathy Baummer Department of Electrical Engineering George Washington University Washington D. C., 20052	
Professor Edward Landesman Department of Mathematics University of California Santa Cruz, CA 95064	
Professor Aihua Shaker Department of Mathematics Naval Postgraduate School Monterey, CA 93943	(10)